# Exercise 1.1.3

(a) Show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

(b) By direct addition $\sum_{n=2}^{100,000} [n(\ln n)^2]^{-1} = 2.02288$. Use Eq. (1.9) to make a five-significant-figure estimate of the sum of this series.

TYPO: Actually, $\sum_{n=2}^{100,000} [n(\ln n)^2]^{-1} = 2.0228839425785075\ldots$.

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**Solution**

## Part (a)

The summand can be integrated, so use the Cauchy integral test to prove convergence. Let

$$f(x) = \frac{1}{x(\ln x)^2}.$$

Both $x$ and $\ln x$ are continuous functions on $[2, \infty)$, so $x(\ln x)^2$ and $[x(\ln x)^2]^{-1}$ are as well. Calculate the first derivative of $f(x)$.

$$f'(x) = \frac{d}{dx}\left[\frac{1}{x(\ln x)^2}\right] = -\frac{2 + \ln x}{x^2(\ln x)^3}$$

Since $\ln x$ is positive and monotonically increasing on $[2, \infty)$, $f'(x) < 0$ on $[2, \infty)$, which means $f(x)$ is monotonically decreasing. $f(x)$ is also positive on $[2, \infty)$. The conditions to use the integral test are satisfied; now calculate the corresponding definite integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2}$$

by substituting $u = \ln x$ ($du = dx/x$).

$$\int_{\ln 2}^{\infty} \frac{du}{u^2}$$

$$\left(-\frac{1}{u}\right)\bigg|_{\ln 2}^{\infty}$$

$$\frac{1}{\ln 2}$$

The definite integral converges, so

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by the Cauchy integral test.

## Part (b)

Equation (1.9) on page 6 gives a lower bound and an upper bound for the remainder of an infinite sum.

$$\int_{N+1}^{\infty} f(x)\,dx \le \sum_{n=N+1}^{\infty} a_n \le \int_{N+1}^{\infty} f(x)\,dx + a_{N+1} \tag{1.9}$$

Since $\sum_{n=2}^{100{,}000}[n(\ln n)^2]^{-1} = 2.0228839425785075\ldots$, set $N = 100{,}000$ in the formula.

$$\int_{100{,}001}^{\infty} f(x)\,dx \le \sum_{n=100{,}001}^{\infty} a_n \le \int_{100{,}001}^{\infty} f(x)\,dx + a_{100{,}001}$$

$$\int_{100{,}001}^{\infty} \frac{dx}{x(\ln x)^2} \le \sum_{n=100{,}001}^{\infty} \frac{1}{n(\ln n)^2} \le \int_{100{,}001}^{\infty} \frac{dx}{x(\ln x)^2} + \frac{1}{(100{,}001)[\ln(100{,}001)]^2}$$

$$\frac{1}{\ln 100{,}001} \le \sum_{n=100{,}001}^{\infty} \frac{1}{n(\ln n)^2} \le \frac{1}{\ln 100{,}001} + \frac{1}{(100{,}001)[\ln(100{,}001)]^2}$$

Add $\sum_{n=2}^{100{,}000} \frac{1}{n(\ln n)^2}$ to all sides.

$$\sum_{n=2}^{100{,}000} \frac{1}{n(\ln n)^2} + \frac{1}{\ln 100{,}001} \le \sum_{n=2}^{100{,}000} \frac{1}{n(\ln n)^2} + \sum_{n=100{,}001}^{\infty} \frac{1}{n(\ln n)^2}$$

$$\le \sum_{n=2}^{100{,}000} \frac{1}{n(\ln n)^2} + \frac{1}{\ln 100{,}001} + \frac{1}{(100{,}001)[\ln(100{,}001)]^2}$$

Combine the sums in the middle, and use the fact that $\sum_{n=2}^{100{,}000}[n(\ln n)^2]^{-1} = 2.0228839425785075\ldots$ on the left and right.

$$2.0228839425785075\ldots + 0.08685882093641431\ldots \le \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$\le 2.0228839425785075\ldots + 0.08685889638020762\ldots$$

$$2.1097427635149217\ldots \le \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \le 2.1097428389587156\ldots$$

Therefore, to five significant figures,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \approx 2.1097.$$